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The decomposition of spin-reducing representations of $N = 2$ supersymmetry

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Abstract. We decompose any spin-reducing representation of $N = 2$ supersymmetry into a direct integral of irreducible representations in a unique manner, in which each irrep has standard form. A single real number is required to classify the irreps in the general representations.

1. Introduction

The existence of a barrier at $N = 3$ in the construction of N-extended supersymmetric Yang–Mills (N-SYM) and supergravity (N-SGR) theories was shown recently for 4-SYM and 3- and 4-SGR in four dimensions (Roček and Siegel 1981, Rivelles and Taylor 1981, Taylor 1982a) and more recently in various dimensions $d \leq 11$ for appropriate N-SYM and N-SGR theories (Rivelles and Taylor 1983). There appear to be essentially three ways to avoid this barrier, by: (i) using $N/2$ supersymmetry explicitly to describe the theories, (ii) destroying explicit Lorentz and gauge covariance by choosing the light-cone gauge, or (iii) changing the supersymmetry algebra by the introduction of central charges. In order to construct a powerful enough framework within which the finiteness of N-SGR can be analysed to all orders, it appears necessary to adopt path (iii) above since (i) will only allow finiteness to be proved up to $(N-1)$ loops (Grisaru and Siegel 1982) and not to all orders by (ii) (Taylor 1982b). Path (iii) has been advocated by one of us (Taylor 1982c) and various candidate Lagrangians for $N = 4$, 6 and 8 SGR presented at the linearised level in which central charges play a crucial role (Taylor 1981).

The particular feature possessed by the use of central charges to avoid the $N = 3$ barrier is that certain irreducible representations (irreps) of the SUSY algebra can arise which have only half the maximum spin of that possessed by the SUSY algebra without central charges. This 'spin reduction' property, known several years ago (Sohnius 1978), corresponds to the masslessness of SUSY irreps in a higher-dimensional space-time in which the extra dimensions arise as coordinates related to the central charges as their canonical momenta. The occurrence of spin reduction is then only natural as a feature of massless, as compared with massive, irreps of SUSY.

The possibility of using these spin-reducing irreps, with central charges, to construct full nonlinear theories of 4-SYM and N-SGR has been made more feasible by the recent construction of a dynamical theory of these irreps using the extra dimensions in a geometric fashion (Restuccia and Taylor 1983). We have now to determine how this geometry is constrained in order that suitable spin-reducing irreps arise so as to give the nonlinearisation of the linearised theories (Taylor 1981). However, even for $N=2$, the spin-reducing properties do not automatically lead to irreps, but in general to multiplets with an infinite number of components. This has been shown for $N=2$ for a purely scalar theory in which supersymmetry has been neglected (Gorse *et al* 1983). Similar features happen for higher N, though a recent analysis (Bufton and Taylor 1983) has shown how a finite spin-reducing irrep can be constructed for $N=4$ by suitable dimensional reduction from ten dimensions. We will consider here solely the case of $N=2$, where a similar approach to that is not appropriate. We will see that we can define a set of spin-reducing irreps for $N=2$, each with the expected field content (Sohnius 1978), and that any spin-reducing superfield can be decomposed into a direct integral (over a single real parameter) of such irreps. Such an integral representation has been given in a general form (Rands and Taylor 1983), though our analysis of the present case will be more direct, and also allow a precise definition to be given of the representation in this case; such precision was lacking in the original case.

We set out our notation and the essential solution to our problem of decomposition in § 2, and give the direct integral representation in § 3. Section 4 gives a conclusion.

2. Decomposition of multiplets in terms of irreducible ones

Let $\phi(x, \theta, \bar{\theta}, x^5, x^6)$ be an $N=2$ superfield which satisfies the spin-reducing condition

$$\not{p}^{\alpha}_{\dot{\alpha}} D^i_{\alpha}\phi = z^{ij}\bar{D}_{\dot{\alpha}j}\phi, \qquad \not{p}^{\dot{\alpha}}_{\alpha}\bar{D}_{\dot{\alpha}i}\phi = \bar{z}_{ij}D^j_{\alpha}\phi, \tag{2.1}$$

where $z^{ij} = \varepsilon^{ij}[\partial_{x^5} + \mathrm{i}\,\partial_{x^6}]$, with $1 \leqslant i, j \leqslant 2$. In general ϕ may have external indices which we have not written. It is a well known result (Taylor 1980) that if (2.1) is satisfied then only Q^i_{α} is needed as the spinor generator of the Poincaré superalgebra. This is the way in which spin reducing follows and it is crucially related with the possibility of avoiding the $N=3$ barrier (Rivelles and Taylor 1983).

All the components $\bar{D}^{(m)}_{\dot{\alpha}j}\phi|_{\theta=\bar{\theta}=x^5=x^6=0}$ of ϕ can be expressed in terms of $D^{(m)i}_{\alpha}\phi|_{\theta=\bar{\theta}=x^5=x^6=0}$ and their derivatives with respect to x^5, x^6 at $\theta=\bar{\theta}=x^5=x^6=0$. Moreover, (2.1) yields

$$z^{ij}\bar{z}_{jk}\phi = p^2\delta^i_k\phi \tag{2.2}$$

which restricts the independent data needed to describe the x^5, x^6 evolution. In the case of one real central charge (2.2) restricts these data solely to ϕ and $\partial_5\phi$. In the general case we have an infinite set of initial data (Gorse *et al* 1983). The superfield $\phi(x, \theta, \bar{\theta}, x^5, x^6)$ which satisfies (2.1) is therefore expected to be highly reducible and we are interested in its decomposition in terms of irreducible multiplets.

In order to isolate an irreducible representation from ϕ satisfying (1) we need two different kinds of constraints. One set of constraints relates the different components $D^m\phi|_{\theta=\bar{\theta}=x^5=x^6=0}$, while the other set of constraints must relate the derivatives of these quantities with respect to x^5 and x^6 in terms of the finite independent data for the x^5, x^6 evolution. In the case of the hypermultiplet ϕ_j the first set of constraints is given by

$$D_{\alpha(i}\phi_{j)} = 0, \tag{2.3a}$$

which implies that all the components of $D^m\phi|_{\theta=\bar{\theta}=x^5=x^6=0}$ can be written in terms of $\phi_j|_{\theta=\bar{\theta}=x^5=x^6=0}$, $D_{\alpha i}\phi^i|_{\theta=\bar{\theta}=x^5=x^6=0}$ and the derivatives with respect to x^5 and x^6 of these independent components. The spin-reducing condition (2.1) yields

$$\bar{D}_{\dot\alpha(i}\phi_{j)} = 0, \qquad (2.3b)$$

and (2.3*a*) and (2.3*b*) are the usual defining conditions for the hypermultiplet. (2.3) does not define an irreducible multiplet, as we have noted before, and we need further constraints which will restrict the x^5, x^6 evolution. We are interested in finding the general requirement which, together with (2.3), yields the irreducibility of the hypermultiplet. Moreover, we are going to prove that, given a general $N = 2$ superfield satisfying the spin-reducing condition (2.1), it can always be decomposed in an unique way in terms of the one-parameter set of superfields $\omega_\varphi(x, \theta, \bar\theta, x^5, x^6)$ defined by the covariant condition, in addition to equation (2.2),

$$\partial_5\omega_\varphi = \tan\varphi\ \partial_6\omega_\varphi. \qquad (2.4)$$

(2.3) and (2.4) for a fixed φ define the irreducible hypermultiplet with one central charge in the φ direction in the x^5, x^6 plane.

3. Superposition of ω_φ modes

For $\varphi = 0$, (2.4) defines the central charge in the x^6 direction, while for $\varphi = \pi/2$, (2.4) defines one central charge in the x^5 direction. In general we may consider the following change of variables:

$$y_5 = \sin\varphi\, x^5 + \cos\varphi\, x^6, \qquad y_6 = -\cos\varphi\, x^5 + \sin\varphi\, x^6. \qquad (3.1)$$

If ω_φ satisfies (2.4) then

$$\partial_{y_6}\omega_\varphi = 0, \qquad \partial^2_{y_5}\omega_\varphi = p^2\omega_\varphi, \qquad (3.2)$$

which define a multiplet with only one central charge in the y_5 direction. We have thus shown that the condition (2.4) always defines a multiplet with only one central charge. Let us now consider a general solution ϕ of (2.1); it automatically satisfies (2.2). It is a well known result that the general solution of (2.2) can be written as a superposition of modes ϕ_φ,

$$\phi = \sum_\varphi \phi_\varphi(p, \theta, \bar\theta, x^5, x^6), \qquad (3.3a)$$

where

$$\phi_\varphi = v_\varphi(p, \theta, \bar\theta, x^5)w_\varphi(p, \theta, \bar\theta, x^6). \qquad (3.3b)$$

The expansion (3.3) follows from the differentiability of the solutions of an elliptic equation $Lu = 0$ in terms of that of the coefficients of the elliptic operator L (Hopf 1931, Douglis and Nirenberg 1955) and the Weirstrass approximation theorem. We obtain from (2.2)

$$v''_\varphi/v_\varphi + w''_\varphi/w_\varphi = p^2, \qquad (3.4)$$

where v'_φ and w'_φ indicate derivatives with respect to x^5 and x^6 respectively. The first term in (3.4) is a function of x^5 and the second of x^6 only, consequently we must have

$$v''_\varphi/v_\varphi = c_\varphi p^2, \qquad w''_\varphi/w_\varphi = (1 - c_\varphi)p^2, \qquad (3.5a, b)$$

where c_φ is independent of x^5 and x^6. The range of c_φ is $(-\infty, +\infty)$. The general solution of (3.5*a*) is a linear combination of the general solutions, v_φ^+ and v_φ^-, of

$$v_\varphi^{+\prime}/v_\varphi^+ = c_\varphi^{1/2}|p|, \qquad v_\varphi^{-\prime}/v_\varphi^- = c_\varphi^{1/2}|p|. \tag{3.6}$$

Analogously, the general solution of (3.5*b*) is a linear combination of the general solutions, w_φ^+ and w_φ^-, of

$$w_\varphi^{+\prime}/w_\varphi^+ = (1-c_\varphi)^{1/2}|p|, \qquad w_\varphi^{-\prime}/w_\varphi^- = -(1-c_\varphi)^{1/2}|p|. \tag{3.7}$$

Therefore, ϕ_φ can always be expressed as

$$\phi_\varphi = \{\phi_\varphi^{++} + \phi_\varphi^{--}\} + \{\phi_\varphi^{+-} + \phi_\varphi^{-+}\}, \tag{3.8}$$

where $\phi_\varphi^{++} \equiv v_\varphi^+ w_\varphi^+$, $\phi_\varphi^{--} \equiv v_\varphi^- w_\varphi^-$, $\phi_\varphi^{+-} \equiv v_\varphi^- w_\varphi^-$, $\phi_\varphi^{-+} \equiv v_\varphi^- w_\varphi^+$. We notice that

$$\begin{aligned} &\partial_5\phi_\varphi^{++} = b_\varphi\,\partial_6\phi_\varphi^{++}, && \partial_5\phi_\varphi^{--} = b_\varphi\,\partial_6\phi_\varphi^{--},\\ &\partial_5\phi_\varphi^{+-} = -b_\varphi\,\partial_6\phi_\varphi^{+-}, && \partial_5\phi_\varphi^{-+} = -b_\varphi\,\partial_6\phi_\varphi^{-+}, \end{aligned} \tag{3.9}$$

where $b_\varphi = c_\varphi^{1/2}/(1-c_\varphi)^{1/2}$, and

$$\begin{aligned} &\partial_5^2\phi_\varphi^{++} = c_\varphi p^2\phi_\varphi^{++}, && \partial_5^2\phi_\varphi^{--} = c_\varphi p^2\phi_\varphi^{--},\\ &\partial_5^2\phi_\varphi^{+-} = (1-c_\varphi)p^2\phi_\varphi^{+-}, && \partial_5^2\phi_\varphi^{-+} = (1-c_\varphi)p^2\phi_\varphi^{-+}. \end{aligned} \tag{3.10}$$

Without loss of generality we can always take $b_\varphi = \tan\varphi$, consequently $c_\varphi = \sin^2\varphi$.

(3.9) and (3.10) show that ϕ_φ^{++} and ϕ_φ^{--} are the two independent modes which define one central charge in the φ direction while ϕ_φ^{+-}, ϕ_φ^{-+} are the two independent modes which define one central charge in the $-\varphi$ direction.

We have thus proved that ϕ_φ has two central charges, one in the $+\varphi$ direction and the other in the $-\varphi$ direction. Both central charges are defined by the irreducible condition (2.4). Therefore we can always express $\phi(p, \theta, \bar\theta, x^5, x^6)$ as

$$\phi(p, \theta, \bar\theta, x^5, x^6) = \pi^{-1}\int_{-\pi/2}^{+\pi/2} \omega_\varphi(p, \theta, \bar\theta, x^5, x^6)\,\mathrm{d}\varphi, \tag{3.11}$$

where ω_φ satisfies (2.4).

We note that the different modes ω_φ in (3.11) are not mixed by SUSY and/or central charge transformations; this fact is a direct consequence of the covariance of (2.4) under both transformations. The uniqueness of the decomposition (3.11) arises trivially from the independence of the cylindrical exponentials in the x^5–x^6 function space. We may write (3.11) in terms of polar coordinates ρ, θ in the x^5, x^6 plane as

$$\phi = \pi^{-1}\int_{-\pi/2}^{+\pi/2} \{\exp[|p|\rho\cos(\theta-\varphi)]A(\varphi) + \exp[-|p|\rho\cos(\theta-\varphi)]B(\varphi)\}\,\mathrm{d}\varphi,$$

which is the usual decomposition of ϕ in terms of cylindrical exponentials. The uniqueness of this decomposition is well known.

We also remark the fact that this representation makes precise the general integral decomposition of degenerate spin-reducing representations given earlier (Rands and Taylor 1983).

We may now consider the transformation properties of ϕ under SUSY transformations. From (3.11) we obtain

$$\delta\phi = \pi^{-1}\int_{-\pi/2}^{+\pi/2} (\xi Q + \bar\xi\bar Q)\omega_\varphi\,\mathrm{d}\varphi.$$

In particular the transformation laws for the irreducible hypermultiplet with one central charge in the $-\varphi$ direction are the usual ones

$$\delta A_i = \xi_i \psi + \bar{\xi}_i \bar{\varphi}, \qquad \delta \psi_\alpha = 2\mathrm{i}\xi_{\alpha i} E^i + 2\mathrm{i}(\not{\partial}\bar{\xi})^i_\alpha A_i,$$

$$\delta \bar{\varphi}_\alpha = 2\mathrm{i}(\xi \not{\partial})_{\alpha i} A^i - 2\mathrm{i}\bar{\xi}^i_\alpha E_i, \qquad \delta E_i = -\xi^\alpha_i (\not{\partial} \in \bar{\varphi})_\alpha - (\varepsilon \not{\partial} \bar{\xi})^\alpha_i \psi_\alpha,$$

but with

$$E_i = (1 + \mathrm{i} \tan^{-1} \varphi) F_i, \qquad F_i \equiv \partial_{x^5} A_i |_{x^5 = x^6 = 0}.$$

4. Conclusions

One of the important features of our result is the appearance of a new parameter describing the direction in the x^5–x^6 plane along which the central charge must lie for a given irrep. We note that new parameters have already been discovered in the decomposition of spin-reducing $N = 4$ multiplets (Bufton and Taylor 1983). The parameter we have now obtained is of a different character from those obtained previously. Those arose from the additional constraints

$$\partial_i^2 = a_i \Box \qquad (5 \leqslant i \leqslant 10) \tag{4.1}$$

with $\Sigma_{i=5}^{10} a_i = 1$. We may still choose complex central charges along each of these six directions, and each of these gives reducible representations, which are then decomposable by our present analysis.

The new parameter, for $N = 2$, or the set of six new parameters for $N = 4$ (or 7 for $N = 8$), will enter in the construction of N-SYM or N-SGR by way of suitable field strength or torsion constraints. For $N = 2$ the parameter φ would only be relevant if two different spin-reducing central charge irreps were required to construct $N = 2$ SGR. Due to the fact that there is only one vector field (the Maxwell field) available to gauge local central charge transformations, such a possibility cannot occur. For $N = 4$ SGR there are six physical gauge vectors, so that we can envisage a form of 4-SGR with multiplets with three complex off-shell central charges. Torsion constraints might be expected to contain the three angles φ_i. There is the alternative possibility of having multiplets with six real central charges, with no overall internal symmetry (as compared with the overall SU(2) symmetry of the previous case). We cannot decide between these two cases without a careful analysis of solutions to torsion constraints. We propose to present this elsewhere (Hassoun *et al* 1983).

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